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Solutions of Podolsky's electrodynamics equation in the first-order formalism

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Abstract
The Podolsky generalized electrodynamics with higher derivatives is formulated in the first-order formalism. The first-order relativistic wave equation in the 20-dimensional matrix form is derived. We prove that the matrices of the equation obey the Petiau–Duffin–Kemmer algebra. The Hermitianizing matrix and Lagrangian in the first-order formalism are given. The projection operators extracting solutions of field equations for states with definite energy–momentum and spin projections are obtained, and we find the density matrix for the massive state. The 13×13-matrix Schrödinger form of the equation is derived, and the Hamiltonian is obtained. Projection operators extracting the physical eigenvalues of the Hamiltonian are found.

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1. Introduction

There is currently a renewal of interest in higher derivative (HD) field theories. HD field equations appear in many models such as renormalizable quantum gravity [1], Podolsky's generalized electrodynamics [2], the Lee–Wick model [3] and others. One of the reasons to consider HD theories is to improve renormalization properties of theories and to remove ultraviolet divergences [4]. However, HD models suffer some difficulties connected with the presence of ghosts. These can lead to the violation of unitarity [5, 6]. Nevertheless, in some HD models these problems with negative probabilities and S-matrix unitarity can be avoided [7]. Also, the quadratic divergence associated with the Higgs mass was removed in the HD Lee–Wick standard model [8], that solves the hierarchy problem. Extensions of the minimal standard model to new physics are justified until observations at the large hadron collider (LHC) are analyzed.

It is well known that in classical electrodynamics the electromagnetic mass is infinite and, therefore, there are infinities associated with a point particle. One of the ways to solve this problem in classical theory is to use the Lorentz invariant regularization of the Maxwell equations at short distances. With the help of an appropriate cutoff the point particle limit can be achieved. This programme was realized in Podolsky's electrodynamics. Firstly the interest to Podolsky's electrodynamics was due to the finiteness of the theory: the electromagnetic energy of a point charge is finite contrarily to ordinary electrodynamics. If distances are much greater than a cutoff, Podolsky's electrodynamics converts into Maxwell's electrodynamics. The solution to Poisson equation for the potential corresponding to a point charge e, located at the origin, in Podolsky's electrostatics is given by [2]

$$\varphi = \frac{e}{4\pi r}(1 - e^{-r/a}),$$

where a is a new parameter of the theory with the dimension of the length playing the role of the cutoff. This potential becomes the Coulomb potential at distances much bigger than a. At $r \to 0$, the potential φ approaches the finite value $e/4\pi a$. The energy of the field for a point charge is also finite in the hole space. Thus, the electrostatic energy can be considered as the regularized electromagnetic mass of a point charge. It was shown [9] that higher derivative terms in Podolsky's equations suppress unphysical runaway solutions with exponentially growing acceleration of the Abraham–Lorentz equation. There are no unwanted solutions if the cutoff is greater than half of the electron classical radius. The upper bound on the parameter is of the order $a \sim 10^{-16}$ cm [9], i.e. the same as the Compton wavelength of the neutral Z-boson. Classical Maxwell's electrodynamics is not valid at small distances and time intervals due to quantum effects. It was also mentioned in [10] that in the framework of non-relativistic quantum theory, a natural cutoff of order of the electron Compton wavelength is effectively appeared by QED processes in close analogy with the classical theory of extended charges. Thus, one may treat the classical Podolsky's electrodynamics as an effective theory where a cutoff introduced, a, is due to the quantum processes at small distances (large momentum). If distances are larger than a, the classical regime begins.

At the same time although QED describes all experimental data well, there are some internal difficulties with the regularization [11]. We mention infrared catastrophe: when the average number of photons $\overline{n} \to \infty$, then the matrix element $|\langle 0 \text{ out}|0 \text{ in}\rangle| \to 0$, and it is impossible to construct the 'out' Fock space from the 'in' space, and impossible to find the unitary operator S [11]. The authors [12] wrote: 'There is an alternative possibility to avoid infrared divergences. We give the photon a small mass μ. This will cut off the low-energy region since now $k^0 > \mu$ and therefore remove the infrared divergence.... The infrared divergences of quantum electrodynamics are essentially classical'. We continue with the citation [13]: 'Another aspect of infrared singularities is related to the long-range character of the Coulomb forces. The latter induces an infinite phase shift on the scattered plane waves. To prevent it, we may introduce a screening factor which in a *consistent theory* would be related to the fictitious photon mass μ'. Thus, in QED the cutoff is introduced 'by hands' as for small distances (to remove ultraviolet divergences) as well as for large distances (to avoid infrared catastrophe). Therefore, one may consider naturally to extend classical Podolsky's electrodynamics on the quantum level where the cutoff is appeared due to the presence of higher derivatives. Anyway, different aspects of Podolsky's electrodynamics, in our opinion, have a definite theoretical interest.

Some features of Podolsky's electrodynamics were investigated in [14–17]. The goal of this paper is to formulate Podolsky's electrodynamics equation in the form of the first-order relativistic wave equation and to obtain solutions in the form of projection matrices.

The paper is organized as follows. In section 2, the third-order field equation is discussed. We derive the first-order relativistic wave equation for Podolsky's electrodynamics in the 20-dimensional matrix form. The Hermitianizing matrix and the Lagrangian in the matrix form are found in section 3. The projection operators extracting solutions of field equations for definite energy and spin states of particles are obtained in section 4. We find the density matrix for the massive state. In section 5, the 13×13-matrix Schrödinger form of the equation is derived, and the Hamiltonian is obtained. Solutions of this equation are found in the form of projection operators. The results are discussed in section 6. In appendix A, we consider the first-order wave equation in the presence of the charge current density. The Lorentz covariance of the equation is proven. Some useful products of matrices are derived in appendix B. We obtain 'minimal' polynomials of the matrix of the equation for massless and massive states. In appendix C, the 'minimal' polynomial of the Hamiltonian matrix is derived.

The Heaviside units are chosen, $\hbar = c = 1$, and Euclidian metric is used, $x_\mu = (x_m, \mathrm{i}x_0)$. Greek letters range from 1 to 4 and Latin letters range from 1 to 3, and there is a summation on repeated indices.

2. Field equations

2.1. Third-order field equations

The Lagrangian of Podolsky's electrodynamics is given by [2]

$$\mathcal{L}_P = -\frac{1}{2}\left[\frac{1}{2}F_{\mu\nu}^2 + a^2(\partial_\mu F_{\nu\mu})^2\right], \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(\mathrm{i}t))$. The dimensional parameter a can be written as $a = 1/m$, where m is the mass parameter. The Euler–Lagrange equations follow from equation (1):

$$\left(\partial_\alpha^2 - m^2\right)\partial_\mu F_{\nu\mu} = 0. \tag{2}$$

The Lagrangian (1) and the equation of motion (1) are gauge invariant under the $U(1)$-group. We can represent equation (2), in the momentum space, as the matrix equation

$$(p^2 + m^2)(p^2 - p \cdot p)A = 0, \tag{3}$$

where $A = \{A_\mu\}$, the matrix-dyad $p \cdot p$, with matrix elements $(p \cdot p)_{\mu\nu} = p_\mu p_\nu$, is introduced and the four-momentum is $p_\mu = (\mathbf{p}, \mathrm{i}p_0)$. The matrix $M = p^2 - p \cdot p$ obeys the minimal polynomial $M(M - p^2) = 0$, so that the eigenvalues of the matrix M are zero and p^2. Thus, equation (3) leads to the dispersion equation

$$p^2(p^2 + m^2) = 0. \tag{4}$$

Equation (4) shows that there are massless and massive states in the spectrum. The propagator of fields is given by

$$\frac{m^2}{p^2(p^2 + m^2)} = \frac{1}{p^2} - \frac{1}{p^2 + m^2}. \tag{5}$$

The first term in equation (5) is the propagator of the photon massless field, and the second term corresponds to the propagator of the massive state of the field. A 'wrong' sign $(-)$ in equation (5) indicates that the massive field state is a ghost. As a result, the massive field state gives the negative contribution to the energy [2], and the classical Hamiltonian is unbounded. To have the positive eigenvalues of the Hamiltonian in the second quantized theory, one has to introduce the indefinite metric. The commutation relations for creation, annihilation operators of the massive state have the wrong sign $(-)$ [2]. The Hilbert space of states is the direct sum of the two subspaces H_p and H_n with positive (H_p) and negative (H_n) square norms. The massless states correspond to a positive square norm, and the massive state to a negative square norm. The transitions between the two subspaces H_p and H_n break the unitarity of the theory.

But if the mass $m \to \infty$ such transitions are forbidden and the unitarity is recovered. Thus, a ghost can be removed in the theory at large m. This procedure is similar to the Pauli–Villars regularization of Feynman diagrams. Therefore, there is a physical sense of the Podolsky theory. We can also argue (similar to Lee–Wick model [8]) that there is no problem with unitarity if the massive photon decays to ordinary fermions through its couplings and is not in the spectrum.

2.2. First-order field equations

Now we reformulate the third-order field equation (2) in the form of first-order relativistic wave equation. Let us consider the system of first-order equations

$$\partial_\mu \psi_{\nu\mu} + m\widetilde{\psi}_\nu = 0, \tag{6}$$

$$\partial_\nu \psi_\mu - \partial_\mu \psi_\nu + m\psi_{\mu\nu} = 0, \tag{7}$$

$$\partial_\mu \widetilde{\psi}_{\nu\mu} + m\widetilde{\psi}_\nu = 0, \tag{8}$$

$$\partial_\nu \widetilde{\psi}_\mu - \partial_\mu \widetilde{\psi}_\nu + m\widetilde{\psi}_{\mu\nu} = 0, \tag{9}$$

where

$$\psi_\mu = mA_\mu, \qquad \psi_{\mu\nu} = F_{\mu\nu}, \qquad \widetilde{\psi}_\mu = \frac{1}{m}\partial_\nu F_{\nu\mu}, \qquad \widetilde{\psi}_{\mu\nu} = \frac{1}{m^2}\partial_\alpha^2 F_{\mu\nu}. \tag{10}$$

After replacing $\widetilde{\psi}_\nu$ from equation (6) and $\widetilde{\psi}_{\mu\nu}$ from equation (9) into equation (8), one obtains equation (2). Equation (7) is the usual equation for the potentials. Thus, we claim that the system of first-order equations (6)–(9) is equivalent to the third-order equation (2). Let us introduce the 20-dimensional wavefunction

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi_\mu(x) \\ \psi_{\mu\nu}(x) \\ \widetilde{\psi}_\mu(x) \\ \widetilde{\psi}_{\mu\nu}(x) \end{pmatrix} \qquad (A = \mu, [\mu\nu], \widetilde{\mu}, \widetilde{[\mu\nu]}), \tag{11}$$

where $\psi_{[\mu\nu]}(x) = \psi_{\mu\nu}(x)$, $\psi_{\widetilde{\mu}}(x) = \widetilde{\psi}_\mu(x)$ and $\psi_{\widetilde{[\mu\nu]}}(x) = \widetilde{\psi}_{\mu\nu}(x)$. The function $\Psi(x)$ represents the direct sum of two four-vectors $\psi_\mu(x)$, $\widetilde{\psi}_\mu(x)$, and two antisymmetric tensors of the second rank $\psi_{\mu\nu}(x)$, $\widetilde{\psi}_{\mu\nu}(x)$.

We explore the elements of the entire matrix algebra $\varepsilon^{A,B}$ [18, 19] with matrix elements and products:

$$(\varepsilon^{M,N})_{AB} = \delta_{MA}\delta_{NB}, \qquad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \tag{12}$$

where $A, B, M, N = \mu, [\mu\nu], \widetilde{\mu}, \widetilde{[\mu\nu]}$, and generalized Kronecker symbols

$$\delta_{[\mu\nu][\alpha\beta]} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}.$$

The $\varepsilon^{M,N}$ are 20×20-matrices that consist of zeros and only one element is unity where the row M and the column N cross.

With the help of equations (11) and (12), the system of equations (6)–(9) can be represented in the form of the first-order equation

$$\partial_\mu(\varepsilon^{\nu,[\nu\mu]} + \varepsilon^{[\nu\mu],\nu} + \varepsilon^{\widetilde{\nu},\widetilde{[\nu\mu]}} + \varepsilon^{\widetilde{[\nu\mu]},\widetilde{\nu}})_{AB}\Psi_B(x)$$
$$+ m\left(\tfrac{1}{2}\varepsilon^{[\nu\mu],[\nu\mu]} + \varepsilon^{\nu,\widetilde{\nu}} + \varepsilon^{\widetilde{\nu},\widetilde{\nu}} + \tfrac{1}{2}\varepsilon^{\widetilde{[\nu\mu]},\widetilde{[\nu\mu]}}\right)_{AB}\Psi_B(x) = 0. \tag{13}$$

There is a summation over all repeated indices. We define 20-dimensional matrices as follows:

$$\beta_\mu = \beta_\mu^{(1)} + \widetilde{\beta}_\mu^{(1)}, \qquad \beta_\mu^{(1)} = \varepsilon^{\nu,[\nu\mu]} + \varepsilon^{[\nu\mu],\nu}, \qquad \widetilde{\beta}_\mu^{(1)} = \varepsilon^{\widetilde{\nu},\widetilde{[\nu\mu]}} + \varepsilon^{\widetilde{[\nu\mu]},\widetilde{\nu}}, \tag{14}$$

$$P = \frac{1}{2}\varepsilon^{[\nu\mu],[\nu\mu]} + \varepsilon^{\nu,\widetilde{\nu}} + \varepsilon^{\widetilde{\nu},\widetilde{\nu}} + \frac{1}{2}\varepsilon^{\widetilde{[\nu\mu]},\widetilde{[\nu\mu]}}. \tag{15}$$

Taking into account equations (14) and (15), equation (13) takes the form of the first-order relativistic wave equation:

$$(\beta_\mu\partial_\mu + mP)\Psi(x) = 0. \tag{16}$$

The presence of the projection operator P in equation (16) is connected with the fact that there is a massless state in the spectrum [19, 20]. Thus, we reformulated the higher derivative equation (2) in the form of the first-order equation (16). The P is the projection operator, $P^2 = P$ [21] and it is not the Hermitian matrix $P^+ \neq P$. The matrices $\beta_\mu^{(1)}$ and $\widetilde{\beta}_\mu^{(1)}$ are Hermitian matrices and have non-zero components in ten-dimensional subspaces $(\mu, [\mu\nu])$, $(\widetilde{\mu}, \widetilde{[\mu\nu]})$, respectively, and obey the Petiau–Duffin–Kemmer algebra [22, 23] (see also [18, 19]):

$$\beta_\mu\beta_\nu\beta_\alpha + \beta_\alpha\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\alpha + \delta_{\alpha\nu}\beta_\mu. \tag{17}$$

Therefore, the matrix β_μ is the direct sum of two ten-dimensional Petiau–Duffin–Kemmer matrices. The projection operator P 'connects' two ten-dimensional subspaces $(\mu, [\mu\nu])$ and $(\widetilde{\mu}, \widetilde{[\mu\nu]})$. Thus, HD Podolsky's electrodynamics equations lead to 'doubling' the dimension of the Petiau–Duffin–Kemmer algebra representation.

3. The Lorentz covariance and Hermitianizing matrix

Let us prove the Lorentz covariance of equation (16). The Lorentz group transformations of coordinates are given by $x'_\mu = L_{\mu\nu}x'_\nu$, where the Lorentz matrix $L = \{L_{\mu\nu}\}$ satisfies the equation $L_{\mu\alpha}L_{\nu\alpha} = \delta_{\mu\nu}$. Wavefunction (11), under the Lorentz coordinates transformations, becomes

$$\Psi'(x') = T\Psi(x), \tag{18}$$

where the 20×20-matrix T realizes the reducible tensor representation of the Lorentz group. The first-order wave equation (16) is transformed into

$$(\beta_\mu\partial'_\mu + mP)\Psi'(x') = (\beta_\mu L_{\mu\nu}\partial_\nu + mP)T\Psi(x) = 0, \tag{19}$$

where $\partial'_\mu = L_{\mu\nu}\partial_\nu$. We have the Lorentz covariance of equation (16) if equations

$$\beta_\mu T L_{\mu\nu} = T\beta_\nu, \qquad PT = TP \tag{20}$$

hold. The infinitesimal Lorentz matrix is given by

$$L_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}, \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \tag{21}$$

where $\varepsilon_{\mu\nu}$ are six parameters defining rotations and boosts. The matrix T at the infinitesimal Lorentz transformations reads

$$T = 1 + \frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}, \tag{22}$$

where $J_{\mu\nu}$ are the generators of the Lorentz group in the 20-dimensional space. With the aid of equations (21) and (22) (using the smallness of parameters $\varepsilon_{\mu\nu}$), we obtain from equation (20)

$$\beta_\mu J_{\alpha\nu} - J_{\alpha\nu}\beta_\mu = \delta_{\alpha\mu}\beta_\nu - \delta_{\nu\mu}\beta_\alpha, \qquad PJ_{\alpha\nu} = J_{\alpha\nu}P. \tag{23}$$

The Lorentz group generators in the 20-dimensional representation space are given by

$$J_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu$$
$$= \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{[\lambda\mu],[\lambda\nu]} - \varepsilon^{[\lambda\nu],[\lambda\mu]} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}} + \varepsilon^{\widetilde{[\lambda\mu]},\widetilde{[\lambda\nu]}} - \varepsilon^{\widetilde{[\lambda\nu]},\widetilde{[\lambda\mu]}}, \quad (24)$$

and obeys equation (23). Thus, we have proved the Lorentz covariance of first-order wave equation (16). In appendix A, we generalize equations considered in the case of field equations with the source. It is easy to verify with the help of equation (12) that the generators (24) obey the usual commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha}J_{\mu\beta} + \delta_{\mu\beta}J_{\nu\alpha} - \delta_{\nu\beta}J_{\mu\alpha} - \delta_{\mu\alpha}J_{\nu\beta}. \quad (25)$$

The Hermitianizing matrix η should satisfy the relations [24]

$$\eta\beta_m = -\beta_m^+\eta^+, \qquad \eta\beta_4 = \beta_4^+\eta^+ \qquad (m = 1, 2, 3). \quad (26)$$

We find

$$\eta = \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2}\varepsilon^{[mn],[mn]} + \varepsilon^{\widetilde{m},\widetilde{m}} - \varepsilon^{\widetilde{4},\widetilde{4}} + \varepsilon^{\widetilde{[m4]},\widetilde{[m4]}} - \frac{1}{2}\varepsilon^{\widetilde{[mn]},\widetilde{[mn]}}. \quad (27)$$

The matrix η is the Hermitian matrix, $\eta^+ = \eta$ and commutes with the projection operator P:

$$\eta P = P\eta. \quad (28)$$

Consider the 'conjugated' wavefunction

$$\overline{\Psi}(x) = \Psi^+(x)\eta = (\psi_\mu, -\psi_{\mu\nu}, \widetilde{\psi}_\mu, -\widetilde{\psi}_{\mu\nu}), \quad (29)$$

and $\Psi^+(x)$ is the Hermitian conjugated wavefunction. We took into account that for neutral fields, (ψ_m, ψ_0) are real variables. Thus, the relativistically invariant bilinear form is $\overline{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x)$. Then, we obtain from equation (16) the 'conjugated' equation

$$\overline{\Psi}(x)(\beta_\mu\overleftarrow{\partial}_\mu - mP^+) = 0. \quad (30)$$

Formally, one can construct the Lagrangian

$$\mathcal{L} = -\frac{1}{2}[\overline{\Psi}(x)(\beta_\mu\partial_\mu + mP)\Psi(x) - \overline{\Psi}(x)(\beta_\mu\overleftarrow{\partial}_\mu - mP^+)\Psi(x)]. \quad (31)$$

By varying the action $S = \int \mathrm{d}^4x\,\mathcal{L}$, corresponding to the Lagrangian (31), we obtain equations of motion (16) and (30). One can check using equations (26) and (29) that the Lagrangian $\mathcal{L}$ is the real function, $\mathcal{L}^* = \mathcal{L}$. In addition, for neutral fields the equality

$$\overline{\Psi}(x)P^+\Psi(x) = \overline{\Psi}(x)P\Psi(x) \quad (32)$$

is valid although $P^+ \neq P$. If one wants to consider charged fields (not photon fields), then the electric current density is given by

$$J_\mu(x) = \frac{\mathrm{i}}{m^3}\left[(\partial_\rho F^*_{\rho\nu})\partial_\alpha^2 F_{\nu\mu} - \left(\partial_\alpha^2 F^*_{\nu\mu}\right)(\partial_\rho F_{\rho\nu})\right], \quad (33)$$

where the complex conjugation $*$ does not act on the metric imaginary unit. Using equations of motion (2), one can verify that electric current is conserved, $\partial_\mu J_\mu(x) = 0$. The electric current density (33) can be cast into the matrix form

$$J_\mu(x) = \mathrm{i}\overline{\Psi}(x)\widetilde{\beta}_\mu^{(1)}\Psi(x). \quad (34)$$

It follows from equation (33) that for the neutral (photon) fields, the electric current density vanishes, $J_\mu(x) = 0$, as it should be.

4. The mass and spin projection operators

Let us consider solutions to equation (16) with definite energy and momentum. In the momentum space, equation (16) becomes

$$\Lambda\Psi(p) = 0, \qquad \Lambda = \mathrm{i}\hat{p} + mP, \qquad \hat{p} = \beta_\mu p_\mu, \tag{35}$$

where p_μ is a four-momentum $p_\mu = (\mathbf{p}, \mathrm{i}p_0)$. Let us consider the massive state, $p^2 = -m^2$. For this case, the 20-dimensional matrix Λ obeys the equation (see (B.5) in appendix B)

$$\Lambda(\Lambda - m)(\Lambda - 2m)(\Lambda^2 - m\Lambda - m^2) = 0. \tag{36}$$

From equation (36), we find the solution to equation (35) in the form of the matrix

$$\Pi = N(\Lambda - m)(\Lambda - 2m)(\Lambda^2 - m\Lambda - m^2), \tag{37}$$

where N is a normalization constant, so that $\Lambda\Pi = 0$. This means that every column of the matrix Π is the solution to equation (35). The requirement that Π is the projection operator, $\Pi^2 = \Pi$, leads to the normalization constant $N = -1/(2m^4)$ [21]. The projection operator (37) extracts solutions to equation (35) for definite energy and momentum corresponding to the massive state.

With the help of the Lorentz group generators (24), we obtain the spin operator (see [21])

$$\sigma_p = -\frac{\mathrm{i}}{2|\mathbf{p}|}\epsilon_{abc}p_a J_{bc} = -\frac{\mathrm{i}}{|\mathbf{p}|}\epsilon_{abc}p_a\beta_b\beta_c. \tag{38}$$

The operator (38) obeys the 'minimal' matrix equation

$$\sigma_p(\sigma_p - 1)(\sigma_p + 1) = 0. \tag{39}$$

In accordance with the general method [21], we obtain the projection operators extracting spin projections ± 1 and 0,

$$S_{(\pm 1)} = \tfrac{1}{2}\sigma_p(\sigma_p \pm 1), \qquad S_{(0)} = 1 - \sigma_p^2, \tag{40}$$

satisfying the relations $S_{(\pm 1)}^2 = S_{(\pm 1)}$, $S_{(\pm 1)}S_{(0)} = 0$, $S_{(0)}^2 = S_{(0)}$.

One may check with the help of equation (12) that the operators (40) commute with the mass projection operator (37). As a result, from equations (37) and (40), we find the projection operators

$$\Delta_{\pm 1} = \Pi S_{(\pm 1)}, \qquad \Delta_0 = \Pi S_{(0)} \tag{41}$$

extracting solutions to equation (35) for definite energy–momentum, spin projections ± 1, 0 for states of particles with the mass m. Equation (41) also defines the density matrix for pure spin states. It follows from the 'minimal' polynomial equation (B.4) that for the massless state, $p^2 = 0$, zero eigenvalues of the matrix Λ are degenerated, and therefore it is impossible to construct solutions to equation (35) in the form of the projection operator [21].

5. Quantum mechanical Hamiltonian

Now we obtain the quantum mechanical Hamiltonian from equations (6)–(9). The Schrödinger form of equations has some attractive features because non-dynamical components of the wavefunction are absent. To find the Schrödinger form of equations (6)–(9), we exclude the non-dynamical components. Equations (6)–(9) can be cast in the form of two systems

$$\begin{aligned} &m\psi_{4m} = \partial_4\psi_m - \partial_m\psi_4, \qquad &m\widetilde{\psi}_{4m} = \partial_4\widetilde{\psi}_m - \partial_m\widetilde{\psi}_4,\\ &\partial_4\psi_{m4} + \partial_n\psi_{mn} = -m\widetilde{\psi}_m, \qquad &\partial_4\widetilde{\psi}_{m4} + \partial_n\widetilde{\psi}_{mn} = -m\widetilde{\psi}_m, \end{aligned} \tag{42}$$

$$m\psi_{mn} = \partial_m\psi_n - \partial_n\psi_m, \qquad m\widetilde{\psi}_{mn} = \partial_m\widetilde{\psi}_n - \partial_n\widetilde{\psi}_m, \qquad m\widetilde{\psi}_4 = \partial_m\widetilde{\psi}_{m4}. \tag{43}$$

We can to exclude auxiliary (non-dynamical) components ψ_{mn}, $\widetilde{\psi}_{mn}$, $\widetilde{\psi}_4$ from equation (43). However, ψ_4 cannot be excluded from equation (42). To introduce the evolution of the ψ_4 in time, we use the Lorentz condition $\partial_m\psi_m + \partial_4\psi_4 = 0$. After replacing the non-dynamical components ψ_{mn}, $\widetilde{\psi}_{mn}$, $\widetilde{\psi}_4$ from equation (43) into equation (42), we obtain the following equations:

$$\begin{aligned}
&\mathrm{i}\partial_t\psi_m = m\psi_{m4} - \partial_m\psi_4,\\
&\mathrm{i}\partial_t\psi_4 = \partial_n\psi_n,\\
&\mathrm{i}\partial_t\widetilde{\psi}_m = m\widetilde{\psi}_{m4} - \partial_m\widetilde{\psi}_4,\\
&\mathrm{i}\partial_t\psi_{n4} = m\widetilde{\psi}_n + \frac{1}{m}\left(\partial_m\partial_n\psi_m - \partial_m^2\psi_n\right),\\
&\mathrm{i}\partial_t\widetilde{\psi}_{n4} = m\widetilde{\psi}_n + \frac{1}{m}\left(\partial_m\partial_n\widetilde{\psi}_m - \partial_m^2\widetilde{\psi}_n\right).
\end{aligned} \tag{44}$$

Equation (44) show that 13 components of the wavefunction $\Psi(x)$ possess the evolution in time. Therefore, we introduce the 13-component wavefunction

$$\Phi(x) = \begin{pmatrix} \psi_\mu(x)\\ \psi_{m4}(x)\\ \widetilde{\psi}_m(x)\\ \widetilde{\psi}_{m4}(x)\end{pmatrix}. \tag{45}$$

With the help of the elements of the matrix algebra equation (12), we rewrite equation (44) in the Schrödinger form

$$\mathrm{i}\partial_t\Phi(x) = \mathcal{H}\Phi(x), \tag{46}$$

where the Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} = {}& m(\varepsilon^{n,[n4]} + \varepsilon^{\widetilde{n},\widetilde{[n4]}} + \varepsilon^{[n4],\widetilde{n}} + \varepsilon^{\widetilde{[n4]},\widetilde{n}}) + (\varepsilon^{4,m} - \varepsilon^{m,4})\partial_m\\
&+ \frac{1}{m}\left[(\varepsilon^{[m4],n} + \varepsilon^{\widetilde{[m4]},\widetilde{n}} - \varepsilon^{\widetilde{n},\widetilde{[m4]}})\partial_m\partial_n - (\varepsilon^{[m4],m} + \varepsilon^{\widetilde{[m4]},\widetilde{m}})\partial_n^2\right].
\end{aligned} \tag{47}$$

From the minimal equation (C.6), obtained in appendix C, we find the projection operators extracting states with positive and negative energies for the massless states ($p^2 = 0$)

$$\Sigma_\pm^0 = \pm\frac{(\mathcal{H}\pm|\mathbf{p}|)\mathcal{H}^2(\mathcal{H}^2 - \mathbf{p}^2 - m^2)(\mathcal{H}^2 - 2\mathbf{p}^2 - m^2)}{2|\mathbf{p}|^3m^2(\mathbf{p}^2 + m^2)}, \tag{48}$$

and the massive states ($p^2 = -m^2$)

$$\Sigma_\pm = \mp\frac{(\mathcal{H}\pm p_0)\mathcal{H}^2(\mathcal{H}^2 - \mathbf{p}^2)\left(\mathcal{H}^2 - \mathbf{p}^2 - p_0^2\right)}{2p_0^3m^2\mathbf{p}^2}. \tag{49}$$

Projection operators (48) and (49) obey the following equations:

$$\begin{aligned}
&\left(\Sigma_\pm^0\right)^2 = \Sigma_\pm^0, \qquad \mathcal{H}\Sigma_\pm^0 = \pm p_0\Sigma_\pm^0 \qquad (p_0 = |\mathbf{p}|),\\
&(\Sigma_\pm)^2 = \Sigma_\pm, \qquad \mathcal{H}\Sigma_\pm = \pm p_0\Sigma_\pm \qquad \left(p_0 = \sqrt{|\mathbf{p}|^2 + m^2}\right).
\end{aligned} \tag{50}$$

Projection operators (48) and (49) can be used to construct physical states in the 13-dimensional space of wavefunctions (45).

6. Conclusion

We have formulated Podolsky's generalized electrodynamics equation with higher derivatives in the form of the 20-component first-order relativistic wave equation. This equation describes vector particles possessing the physical massless state and the massive state that is a ghost. To obtain the consistent theory, the mass of the vector state should be very large. One can speculate that the massive vector particles can be described in a gauge-invariant manner by this theory. For the massive state to be the physical state, we have to use the reverse sign in the Lagrangian. Then the Hamiltonian also changes the sign. In this case, however, the massless state becomes the ghost and the question arises: how do we get rid of it? Therefore, the description of massive particles by Podolsky's generalized electrodynamics is questionable. The relativistically invariant bilinear form and the Lagrangian were obtained, and these allow us to use the advantages of the formulation of relativistic wave equations. The density matrix obtained can be used for quantum electrodynamics calculations in the first-order formalism. It should be noted that the Petiau–Duffin–Kemmer form of equations was used in quantum chromodynamics [25], i.e. in non-Abelian theory.

The 13×13-matrix Schrödinger form of the equation is derived, and the Hamiltonian is obtained. We found projection operators extracting the physical eigenvalues of the Hamiltonian. The Schrödinger picture has some advantages by considering field interactions.

Appendix A.

Let us consider the field equation (2) with the source of electromagnetic fields—the charge current density:

$$\left(\partial_\alpha^2 - m^2\right)\partial_\mu F_{\nu\mu} = -m^2 \widetilde{j}_\nu. \tag{A.1}$$

We have introduced the current $\widetilde{j}_\nu$ with the same dimension as in classical electrodynamics. The first-order equations (6), (7) and (9) reman the same but equation (8) is replaced by

$$\partial_\mu \widetilde{\psi}_{\nu\mu} + m\widetilde{\psi}_\nu = \widetilde{j}_\nu(x). \tag{A.2}$$

Then equation (16) becomes

$$(\beta_\mu\partial_\mu + mP)\Psi(x) = P_0 j(x), \tag{A.3}$$

where

$$P_0 = \varepsilon^{\widetilde{\mu},\widetilde{\mu}}, \qquad j(x) = \begin{pmatrix} j_\mu(x) \\ j_{\mu\nu}(x) \\ \widetilde{j}_\mu(x) \\ \widetilde{j}_{\mu\nu}(x) \end{pmatrix}, \tag{A.4}$$

and P_0 is the projection operator, $P_0^2 = P_0$, $P_0^+ = P_0$. The projection operator P_0 extracts only the current $\widetilde{j}_\mu$. Therefore, the currents $j_\mu(x)$, $j_{\mu\nu}(x)$ and $\widetilde{j}_{\mu\nu}(x)$ are not present in the theory and can be put zero. At the Lorentz transformations, $j'(x) = Tj(x)$, and the Lorentz covariance of equation (A.3) follows from equations (20) and (23) and

$$P_0 T = T P_0, \qquad P_0 J_{\mu\nu} = J_{\mu\nu} P_0. \tag{A.5}$$

The Hermitianizing matrix η (27) commutes with P_0, $\eta P_0 = P_0\eta$. Then equation (30) is replaced by

$$\overline{\Psi}(x)(\beta_\mu \overleftarrow{\partial}_\mu - mP^+) = \overline{j}(x)P_0, \tag{A.6}$$

where $\overline{j}(x) = (j_\mu(x), -j_{\mu\nu}(x), \widetilde{j}_\mu(x), -\widetilde{j}_{\mu\nu}(x))$. We obtain the classical limit at $m \to \infty$ $(a \to 0)$ for Maxwellian electrodynamics from equation (A.1):

$$\partial_\mu F_{\nu\mu} = \widetilde{j}_\nu. \tag{A.7}$$

Thus, equation (A.7) is the standard Maxwell equation with the source term.

Appendix B.

With the help of equation (12), we obtain the products of matrices entering equation (35):

$$\hat{p}^3 = p^2\hat{p}, \qquad \hat{p}P + P\hat{p} = P\hat{p}P + \hat{p}, \qquad \hat{p}^2P = P\hat{p}^2, \tag{B.1}$$

$$\hat{p}P\hat{p}^2 = p^2\hat{p}P, \qquad \hat{p}P\hat{p}(1-P) = \hat{p}^2(1-P), \qquad (1-P)\hat{p}^2P = 0. \tag{B.2}$$

Using equations (14), (B.1) and (B.2), one finds

$$\begin{aligned}
&\Lambda(\Lambda - m) = \mathrm{i}mP\hat{p}P - \hat{p}^2,\\
&\Lambda(\Lambda - m)[\Lambda(\Lambda - m)^2 + 2p^2(\Lambda - m) + mp^2] = -\mathrm{i}p^4\hat{p} - mp^2\hat{p}^2P,\\
&\Lambda(\Lambda - m)[\Lambda(\Lambda - m) - m(\Lambda - m) + 2p^2] = \mathrm{i}mp^2\hat{p} - p^2\hat{p}^2 - m^2\hat{p}^2(1-P).
\end{aligned} \tag{B.3}$$

From equations (B.1)–(B.3), we obtain 'minimal' polynomials of the matrix Λ for two states:

$$\Lambda^2(\Lambda - m)^3 = 0, \qquad p^2 = 0, \tag{B.4}$$

$$\Lambda(\Lambda - m)(\Lambda - 2m)(\Lambda^2 - m\Lambda - m^2) = 0, \qquad p^2 = -m^2. \tag{B.5}$$

It should be noted that zero eigenvalues of the matrix Λ for the massless state are degenerated.

Appendix C.

From equation (47), we obtain the Hamiltonian in the momentum space:

$$\begin{aligned}
\mathcal{H} = {}& m(\varepsilon^{n,[n4]} + \varepsilon^{\widetilde{n},\widetilde{[n4]}} + \varepsilon^{[n4],\widetilde{n}} + \varepsilon^{\widetilde{[n4]},\widetilde{n}}) + \mathrm{i}p_m(\varepsilon^{4,m} - \varepsilon^{m,4})\\
&+ \frac{1}{m}[(\varepsilon^{[m4],m} + \varepsilon^{\widetilde{[m4]},\widetilde{m}})\mathbf{p}^2 - (\varepsilon^{[m4],n} + \varepsilon^{\widetilde{[m4]},\widetilde{n}} - \varepsilon^{\widetilde{n},\widetilde{[m4]}})p_mp_n].
\end{aligned} \tag{C.1}$$

Using equation (12), one finds

$$\mathcal{H}^2 - \mathbf{p}^2 = m^2(\varepsilon^{[n4],\widetilde{[n4]}} + \varepsilon^{\widetilde{[n4]},\widetilde{[n4]}} + \varepsilon^{\widetilde{n},\widetilde{n}} + \varepsilon^{n,\widetilde{n}}) + \mathrm{i}mp_n\varepsilon^{4,[n4]} - p_mp_n(\varepsilon^{[n4],[m4]} - \varepsilon^{[n4],\widetilde{[m4]}}), \tag{C.2}$$

$$\begin{aligned}
\mathcal{H}^2 - \mathbf{p}^2 - m^2 = {}& m^2(\varepsilon^{[n4],\widetilde{[n4]}} - \varepsilon^{[n4],[n4]} + \varepsilon^{n,\widetilde{n}} - \varepsilon^{\mu,\mu})\\
&+ \mathrm{i}mp_n\varepsilon^{4,[n4]} - p_mp_n(\varepsilon^{[n4],[m4]} - \varepsilon^{[n4],\widetilde{[m4]}}).
\end{aligned} \tag{C.3}$$

Multiplying equation (C.2) and equation (C.3), we obtain

$$\begin{aligned}
(\mathcal{H}^2 - \mathbf{p}^2)(\mathcal{H}^2 - \mathbf{p}^2 - m^2) = {}& \mathrm{i}m(m^2 + \mathbf{p}^2)p_m(\varepsilon^{4,\widetilde{[m4]}} - \varepsilon^{4,[m4]})\\
&+ (m^2 + \mathbf{p}^2)p_mp_n(\varepsilon^{[n4],[m4]} - \varepsilon^{[n4],\widetilde{[m4]}}).
\end{aligned} \tag{C.4}$$

Squaring equation (C.4), one finds

$$(\mathcal{H}^2 - \mathbf{p}^2)^2(\mathcal{H}^2 - \mathbf{p}^2 - m^2)^2 = \mathbf{p}^2(m^2 + \mathbf{p}^2)(\mathcal{H}^2 - \mathbf{p}^2)(\mathcal{H}^2 - \mathbf{p}^2 - m^2). \tag{C.5}$$

From equation (C.5), we obtain the 'minimal' polynomial of the Hamiltonian

$$\mathcal{H}^2(\mathcal{H}^2 - \mathbf{p}^2)(\mathcal{H}^2 - \mathbf{p}^2 - m^2)(\mathcal{H}^2 - 2\mathbf{p}^2 - m^2) = 0. \tag{C.6}$$

Eigenvalues of the Hamiltonian squared read from equation (C.6) $p_0^2 = 0$, $p_0^2 = \mathbf{p}^2$, $p_0^2 = \mathbf{p}^2+m^2$, $p_0^2 = 2\mathbf{p}^2+m^2$. Thus, there are two physical eigenvalues, $p_0^2 = \mathbf{p}^2$, $p_0^2 = \mathbf{p}^2+m^2$, corresponding to the massless and massive states of the field, and two nonphysical eigenvalues. Equation (C.6) can be used to find projection operators extracting physical states in the Schrödinger picture.

References

[1] Stelle K S 1977 *Phys. Rev.* D **16** 953

[2] Podolsky B 1942 *Phys. Rev.* **62** 68
Podolsky B and Schwed P 1948 *Rev. Mod. Phys.* **20** 40

[3] Lee T D and Wick G C 1969 *Nucl. Phys.* B **9** 209
Lee T D and Wick G C 1970 *Phys. Rev.* D **2** 1033

[4] Thirring W 1950 *Phys. Rev.* **77** 570

[5] Pais A and Uhlenbeck G E 1950 *Phys. Rev.* **79** 145

[6] Heisenberg W 1957 *Nucl. Phys.* **4** 532

[7] Hawking S W and Hertog T 2002 *Phys. Rev.* D **65** 103515

[8] Grinstein B, O'Connell D and Wise M B 2008 *Phys. Rev.* D **77** 025012

[9] Frenkel J 1996 *Phys. Rev.* E **54** 5859
Frenkel J and Santos R B 1999 *Int. J. Mod. Phys.* B **13** 315

[10] Moniz E J and Sharp D H 1974 *Phys. Rev.* D **10** 1133
Moniz E J and Sharp D H 1977 *Phys. Rev.* D **15** 2850

[11] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)

[12] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill) p 172

[13] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill) p 350

[14] Galvao C A P and Pimentel B M 1988 *Can. J. Phys.* **66** 460

[15] Barcelos-Neto J, Galvao C A P and Natividade C P 1991 *Z. Phys.* **52** 559

[16] Accioly A and Mukai H 1997 *Nuovo Cimento* B **112** 1061
Accioly A and Dias M 2004 *Phys. Rev.* D **70** 107705

[17] Cuzinatto R R, de Melo C A M and Pompeia P J 2007 *Ann. Phys.* **322** 1211

[18] Bogush A A and Moroz L G 1968 *Vvedenie v teoriu klassicheskih polei* (Minsk: Nauka i Tekhnika) (in Russian)

[19] Kruglov S I 2001 *Symmetry and Electromagnetic Interaction of Fields with Multi-Spin* (Huntington, NY: Nova Science)

[20] Kruglov S I 2001 *Ann. Fond. Broglie* **26** 725
Kruglov S I 2008 *Can. J. Phys.* **86** 995

[21] Fedorov F I 1959 *Sov. Phys.—JETP* **35** 339 (Engl. Transl.)
Fedorov F I 1958 *Zh. Eksp. Teor. Fiz.* **35** 493

[22] Duffin R J 1939 *Phys. Rev.* **54** 1114

[23] Kemmer H 1939 *Proc. R. Soc.* **173** 91

[24] Gel'fand I M, Minlos R A and Ya Z 1963 *Shapiro, Representations of the Rotation and Lorentz Groups and Their Applications* (New York: Pergamon)

[25] Gribov V 1999 *Eur. Phys. J.* C **10** 71